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A Generalization of Jackson's Inequality

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This note answers a question raised by C. Micchelli about the constants in Jackson's inequality for splines with uniform knots as functions of the degree as well as the number of knots.

THEOREM. *Let $S_{k,n}$ be the space of smooth splines of degree $k-1$ (i.e., $S_{k,n} \subset C^{k-2}$) with n equally spaced knots on the interval* $[0, 1]$. *Then*

$$\inf_{s \in S_{k,n}} \|f-s\|_p \leqslant C \cdot \frac{1}{n+k} \|f'\|_p, \tag{1}$$

where C is an absolute constant less than 10.

For $n=0$, (1) reduces to Jackson's well-known inequality for polynomial approximation. A repeated application of (1) gives

$$\inf_{s \in S_{k,n}} \|f-s\|_p \leqslant C^j \frac{1}{n+k} \cdots \frac{1}{n+k-j+1} \|f^{(j)}\|_p, \qquad k \geqslant j, \tag{2}$$

which shows that in the usual error estimates for spline approximation the constants do not depend on the degree.

To prove (1), recall a result on cardinal spline interpolation [2, 8, 9]:

THEOREM. *Let $I_k f$ be the cardinal spline interpolant to f, that is, a smooth spline of degree $k-1$ with knots at the integers, interpolating f at each $\nu \in \mathbb{Z}$ for even k and at each $\nu + \frac{1}{2}$, $\nu \in \mathbb{Z}$, for odd k. Then*

$$\|f - I_k f\|_{p,\mathbb{R}} \leqslant C_k \|f^{(k)}\|_{p,\mathbb{R}}, \tag{3}$$

where

$$C_k = 4\pi^{-(k+1)} \cdot \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(k+1)}}{(2\nu+1)^{(k+1)}} \leqslant \frac{16}{3} \pi^{-(k+1)}$$

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is the supremum norm of the Euler spline of degree k, the extremal function for the estimate (3).

The estimate (3) was proved in [2, 8] for $p = \infty$. But, since the Peano kernel $K(x, y)$ for cardinal interpolation is symmetric for even k, and for odd k satisfies $K(x, y) = -K(y - \frac{1}{2}, x - \frac{1}{2})$, (3) is valid for $p = 1$, too, and hence, by interpolation theory, for all p.

Proof of (1). Let $f \in W_p^1[0, 1]$, $\|f'\|_p \leqslant 1$, be given and assume f to be periodic with period 1.

One can use Jackson's kernel J_m to approximate f by a trigonometric polynomial of degree at most $2m$ in such a way that [6, p. 37]

$$\|f - J_{\lfloor m/2 \rfloor} * f\|_p \leqslant 2m^{-1}$$

because the L_∞-bound in [6] remains valid for arbitrary p, $1 \leqslant p \leqslant \infty$.

Since $\|J_{\lfloor m/2 \rfloor}\|_1 = 1$, an integration by parts shows that the trigonometric polynomials $t_m := J_{\lfloor m/2 \rfloor} * f$ have uniformly bounded derivatives

$$\|t'_m\|_p \leqslant 1.$$

An application of Bernstein's inequality [1, p. 100] on the interval [0, 1] yields

$$\|t_m^{(k)}\|_p \leqslant (2\pi m)^{k-1}.$$

After extending $t_m^{(k)}$ by 0 outside [0, 1], one may use (3) to obtain, by the usual scaling argument, the estimate

$$\begin{aligned} \|t_m - s\|_p &\leqslant C_k(n+1)^{-k}(2\pi m)^{k-1} \\ &\leqslant \frac{16}{3\pi}((n+1)\pi)^{-k}(2\pi m)^{k-1} \end{aligned}$$

for some $s \in S_{k,n}$. Putting $m = \lfloor (n+1)/2 \rfloor$ and combining the previous estimates one finally ends up with

$$\begin{aligned} \|f - s\|_p &\leqslant \|f - t_m\|_p + \|t_m - s\|_p \\ &\leqslant 4n^{-1} + \frac{16}{3\pi}(n\pi)^{-1} \leqslant 4.6n^{-1}. \end{aligned}$$

The restriction $f(0) = f(1)$ is not essential. In the general case, f is replaced by $f(x) - x\int_0^1 f'(t)\,dt$ and since

$$\left\| f' - \int_0^1 f'(t)\,dt \right\|_p \leqslant 2\|f'\|_p$$

the above estimate implies

$$\inf_{s \in S_{k,n}} \|f - s\|_p \leqslant 9.2 n^{-1}.$$

On the other hand, by Jackson's inequality [5, p. 147], which is easily seen to hold for arbitrary p, $1 \leqslant p \leqslant \infty$,

$$\inf_{s \in S_{k,n}} \|f - s\|_p \leqslant \inf_{s \in S_{k,0}} \|f - s\|_p \leqslant \frac{\pi}{4} k^{-1} \|f'\|_p.$$

Therefore, to prove (1), it suffices to choose C in such a way that

$$\operatorname{Min} \left\{ 9.2 n^{-1}, \frac{4}{\pi} k^{-1} \right\} \leqslant C \frac{1}{n+k}$$

and an easy computation shows that $C = 10$ will do.

By the same method a slightly more general result for splines on a nonuniform mesh can be established.

THEOREM. *Let $S_{k,t}$ be the space of smooth splines on a mesh* $\mathbf{t} = \{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1\}$. *Then*

$$\inf_{s \in S_{k,t}} \|f - s\|_p \leqslant C'((\bar{\Delta})^{-1} + k)^{-j} \|f^{(j)}\|_p, \qquad k \geqslant j, \tag{2'}$$

where C' only depends on j, and $\bar{\Delta} := \max_{0 \leqslant i \leqslant n} (t_{i+1} - t_i)$.

In the proof of (2′) the estimate (3) is replaced by the standard estimate for spline approximation

$$\inf_{s \in Sk,t} \|f - s\|_p \leqslant d^k (\bar{\Delta})^k \|f^{(k)}\|_p, \qquad d \text{ an absolute constant}, \tag{3'}$$

which may be proved by means of quasiinterpolation [3; 4, p. 155, 176; 7].

Remarks. I have made no attempt to determine the best possible value of the constant C and leave it as an open problem to construct directly without use of trigonometric polynomials an approximation process which gives some more precise estimate.

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